Brownian motion and Stochastic Calculus
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Assignment 10

Exercise 1

We fix a standard one-dimensional $(\mathbb{F},\mathbb{P})$–Brownian motion.

1) Show that for any $C^{1,2}$ function $f:[0,+\infty)\times\mathbb{R}\longrightarrow\mathbb{R}$, such that there exists some continuous function $C:[0,+\infty)\longrightarrow[0,+\infty)$ with
$$|\partial_x f(t,x)|\leq C(t)\mathrm{e}^{C(t)|x|},\ (t,x)\in[0,+\infty)\times\mathbb{R}, \tag{0.1}$$
the process $(f(t,B_t))_{t\geq 0}$ will be an $(\mathbb{F},\mathbb{P})$-martingale if and only if
$$\partial_t f(t,x)+\frac{1}{2}\partial^2_{xx}f(t,x)=0,\ (t,x)\in[0,+\infty)\times\mathbb{R}. \tag{0.2}$$

2) in this question, we are looking for functions f of the form
$$f(t,x)=\sum_{i=0}^n\sum_{j=0}^n a_{i,j}t^ix^j,\ (t,x)\in[0,+\infty)\times\mathbb{R},$$
for some integer n and real numbers $(a_{i,j})_{(i,j)\in\{0,\dots,n\}^2}$. Show that the process $f(t,B_t)$ is an $(\mathbb{F},\mathbb{P})$-martingale if and only if the $(a_{0,j})_{j\in\{0,\dots,n\}}$ are arbitrarily fixed and
$$\begin{cases} a_{i,j}=(-1)^i\dfrac{(j+2i)!}{2i!j!}a_{0,j+2i},\ j+2i\leq n,\\ a_{i,j}=0,\ j+2i>n,\end{cases}$$

Exercise 2

Consider, for any $x\in\mathbb{R}^d$, the SDE
$$\mathrm{d}X_t^x=a(X_t^x)\mathrm{d}t+b(X_t^x)\mathrm{d}W_t,\ X_0^x=x,$$
where W is a $\mathbb{R}^m$-valued Brownian motion, $a:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ and $b:\mathbb{R}^d\longrightarrow\mathbb{R}^{d\times m}$ are measurable and locally bounded. We fix a non-empty, bounded open subset U of $\mathbb{R}^d$ and assume that for any $x\in U$, we have with $T_U^x:=\inf\{s\geq 0: X_s^x\notin U\}$, that T_U^x is $\mathbb{P}$-integrable.

Moreover, consider the boundary problem
$$Lu(x)+c(x)u(x)=-f(x),\ \text{for } x\in U,\ u(x)=g(x),\ \text{for } x\in\partial U,$$
where $f\in C_b(U)$, $g\in C_b(\partial U)$, $c\leq 0$ is a uniformly bounded function on $\mathbb{R}^d$, and L is defined by
$$Lf(x):=\sum_{i=1}^d a^i(x)\frac{\partial f}{\partial x^i}(x)+\frac{1}{2}\sum_{(i,j)\in\{1,\dots,d\}^2}\big(bb^\top\big)^{ij}(x)\,\frac{\partial^2 f}{\partial x^i\,\partial x^j}(x).$$

Show that if $u\in C^2(U)\cap C(\bar{U})$ is a solution of the above boundary problem and $(X_t^x)_{t\geq 0}$ is a solution of the SDE for some $x\in U$, then
$$u(x)=\mathbb{E}^{\mathbb{P}}\bigg[g(X^x_{T_U^x})\exp\bigg(\int_0^{T_U^x}c(X_s^x)\mathrm{d}s\bigg)\bigg]+\mathbb{E}^{\mathbb{P}}\bigg[\int_0^{T_U^x}f(X_s^x)\exp\bigg(\int_0^s c(X_r^x)\mathrm{d}r\bigg)\mathrm{d}s\bigg].$$

Exercise 3

Let $(B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion.

1) Show that the SDE

$$X_t = x + \int_0^t \sqrt{1+X_s^2}\mathrm{d}B_s + \frac{1}{2}\int_0^t X_s \mathrm{d}s, \tag{0.3}$$

admits a unique strong solution for all $x \in \mathbb{R}$.

2) Fix $x \in \mathbb{R}$ and $(\beta_t, \gamma_t)_{t\geq 0}$ two independent one-dimensional Brownian motions. Show that

$$Y_t := \exp(\beta_t)\left(x + \int_0^t \exp(-\beta_s) d\gamma_s\right),\ t \geq 0,$$

is well-defined and solves (0.3) fro some well-chosen Brownian motion B. Deduce that for $a := \mathrm{argsinh}(x)$,

$$(Y_t,\ t \geq 0) \overset{\text{(law)}}{=} (\sinh(a + B_t),\ t \geq 0).$$

3) We now go to a slightly more general setting.

a) Show that if the map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^2 diffeomorphism from $\mathbb{R}$, then $\Phi_t := \varphi(B_t)$ satisfies

$$\Phi_t = \varphi(0) + \int_0^t \sigma(\Phi_s)\mathrm{d}B_s + \int_0^t b(\Phi_s)\mathrm{d}s, \tag{0.4}$$

where

$$\sigma(x) := (\varphi' \circ \varphi^{(-1)})(x),\ b(x) := \frac{1}{2}(\varphi'' \circ \varphi^{(-1)})(x).$$

b) Conversely, if $\sigma, b : \mathbb{R} \longrightarrow \mathbb{R}$ are Lispchitz functions with appropriate growth, we know that the SDE (0.4) admits a unique strong solution. Under which conditions on (σ, b) can we solve the system

$$\varphi'(y) = \sigma(\varphi(y)),\ \varphi''(y) = 2b(\varphi(y)),$$

so that the solution of (0.4) is $\Phi_t = \varphi(B_t)$?